Tutorial 8

Aim to give a representation for π as infinite sum

$$(\arctan)'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 + \dots + (-1)^n x^{2n} + \dots$$

Radius of convergence for the above power series is 1.

Hence, $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots \quad \forall\ x \in (-1, 1)$

For $x = 1$, $1 - \frac{1}{3} + \frac{1}{5} + \dots + (-1)^n \frac{1}{2n+1} + \dots$ exists $\therefore$

$$S_m := 1 - \frac{1}{3} + \frac{1}{5} + \dots + (-1)^m \frac{1}{2m+1} \quad . m = 0, 1, 2, \dots$$

$$S_{2n} = 1 - \frac{1}{3} + \frac{1}{5} + \dots + (-1)^{2n} \frac{1}{4n+1} = 1 - \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \dots - \left(\frac{1}{4n} - \frac{1}{4n+1}\right)$$

$\Rightarrow (S_{2n})_{n=0}^{\infty}$ is monotone decreasing sequence

$$S_{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} + \dots + (-1)^{2n+1} \frac{1}{4n+3} = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \dots + \left(\frac{1}{4n+1} - \frac{1}{4n+3}\right)$$

$(S_{2n+1})_{n=0}^{\infty}$ is monotone increasing sequence.

Note also. $\quad 0 \le S_{2n-1} \le S_{2n} \le 1 \quad$ for $n \ge 1$

By MCT, both $(S_{2n})_{n=0}^{\infty}$, $(S_{2n+1})_{n=0}^{\infty}$ converges

They have the same limit because $|S_{2n+1} - S_{2n}| = \frac{1}{4n+3} \to 0$ as $n \to \infty$

One can conclude that $(S_m)_{m=0}^{\infty}$ converges to the same limit.

It is tempted to write $\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} + \dots + (-1)^n \frac{1}{2n+1} + \dots$

But it is not legitimate to do so because we don't know whether

$$\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} + \dots + (-1)^n \frac{1}{2n+1} + \dots$$

or equivalently whether $\lim_{r \to 1} \arctan(r) = 1 - \frac{1}{3} + \frac{1}{5} + \dots + (-1)^n \frac{1}{2n+1} + \dots$

or equivalently whether $\lim_{r \to 1} \left(\sum_{k=0}^{\infty} (-1)^k \frac{r^{2k+1}}{2k+1} \right) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}$

i.e. we don't know whether "$\sum_{k=0}^{\infty} a_k$ exists" implies "$\lim_{r \to 1} \sum_{k=0}^{\infty} a_k r^k = \sum_{k=0}^{\infty} a_k$."

which looks like taking limit termwisely in the series.

It is guaranteed by the following theorem.

Thm : If $\sum_{k=0}^{\infty} a_k$ exists, then $\sum_{k=0}^{\infty} a_k r^k$ exists for $-1<r<1$ and $\lim_{r\to 1^-}\sum_{k=0}^{\infty} a_k r^k = \sum_{k=0}^{\infty} a_k$.

Pf : The first part is related to radius of convergence and we don't discuss it here.

From first part, we can talk about $\lim_{r\to 1^-}\sum_{k=0}^{\infty} a_k r^k$. To check the convergence, we estimate

$$\sum_{k=0}^{\infty} a_k r^k - \sum_{k=0}^{\infty} a_k = \sum_{k=1}^{\infty} a_k (r^k-1) = \sum_{k=1}^{\infty} (r-1)a_k(1+r+\cdots+r^{k-1})$$

$$= (r-1)\sum_{k=1}^{\infty} a_k(1+r+\cdots+r^{k-1})$$

Guess

$$= (r-1)\sum_{n=0}^{\infty} r^n\left(\sum_{m=n+1}^{\infty} a_m\right)$$

for any $N\in\mathbb{N}$

$$= (r-1)\left(\sum_{n=0}^{N} r^n\left(\sum_{m=n+1}^{\infty} a_m\right)\right) + (r-1)\left(\sum_{n=N+1}^{\infty} r^n\left(\sum_{m=n+1}^{\infty} a_m\right)\right)$$

The first term is finite sum, $\to 0$ as $r\to 1^-$

(The second term) is infinite sum with n starting from $N+1$.

$$(r-1)\sum_{n=N+1}^{\infty} r^n\left(\sum_{m=n+1}^{\infty} a_m\right)$$

Since $\sum_{m=0}^{\infty} a_m$ exists, $\left|\sum_{m=n+1}^{\infty} a_m\right|$ is small whenever n is large enough

For N large,

$$\therefore \left|\sum_{n=N+1}^{\infty} r^n\left(\sum_{m=n+1}^{\infty} a_m\right)\right| \le \sum_{n=N+1}^{\infty} r^n\left|\sum_{m=n+1}^{\infty} a_m\right| \le \sum_{n=N+1}^{\infty} r^n\cdot\varepsilon = \varepsilon\cdot\sum_{n=N+1}^{\infty} r^n \le \frac{\varepsilon}{1-r}$$

$\therefore$ The second term is OK whenever N is large independent of r.

First choose N large, decompose into two terms by our guess, the second term is now guaranteed, the first term is OK when $r\to 1^-$ $\therefore$ OK.

Instead of the Guess equality, we can look more carefully. $\sum_{k=1}^{\infty} a_k(1+r+\cdots+r^{k-1}) = \lim_N \sum_{k=1}^{N} a_k(1+r+\cdots+r^{k-1})$

$$= \lim_N \sum_{j=0}^{N-1} r^j\left(\sum_{i=j+1}^{N} a_i\right) = \lim_N\left(\sum_{j=0}^{N-1} r^j b_j(N)\right)$$ where $b_j(N) := \sum_{i=j+1}^{N} a_i$, $b_j(N)\to b_j := \sum_{i=j+1}^{\infty} a_i$.

We can now apply the same technique as above after the Guess equality :

$$\lim_N \sum_{j=1}^{N-1} r^j b_j(N) = \lim_N\left(\sum_{j=1}^{N_0} r^j b_j(N) + \sum_{j=N_0+1}^{N-1} r^j b_j(N)\right)$$ for any $N_0\in\mathbb{N}$.

For j large, $|b_j(N)|$ is small no matter what N you choose, thus the second term

$$\left|\sum_{j=N_0+1}^{N-1} r^j b_j(N)\right| \le \varepsilon\cdot\frac{1}{1-r}$$ for large N_0. Whenever N_0 is chosen, the first term tends to $\sum_{j=1}^{N_0} r^j b_j$ as $N\to\infty$. Multiplying $(r-1)$ and letting $r\to 1^-$, we are done.

$\Rightarrow$ Doesn't matter for the proof, can you find $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n+1}\frac{1}{n}+\cdots$ by the same method ? ☆☆☆ One can estimate Taylor's remainder term of arctan to get the same conclusion.

Tutorial 8

Note that $\frac{1}{1+t^2} = 1 - t^2 + t^4 - \dots + (-1)^{n-1} t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2}$ for $|t|<1$

For $|x| \leq 1$, we have

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + R_{2n}(x) \quad \text{— } (\star)$$

where $R_{2n}(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} dt$. Since $|R_{2n}(x)| \leq \int_0^1 \frac{t^{2n}}{1+t^2} dt \leq \int_0^1 t^{2n} dt = \frac{1}{2n+1}$,

and $(\star)$ holds $\forall n \in \mathbb{N}$, by taking limit $n \to \infty$ on $(\star)$, we get the same conclusion.